The Borel conjecture
(through controlled G-theory)

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Topology is hard. A good idea is to geometrize your problem and exploit the geometry.

- Nikita Selinger

Topological rigidity

Poincare conjecture: if $M^n \simeq S^n$ then $M^n \approx S^n$.

Not true for arbitrary closed manifolds. Lens spaces $L(7,1) \simeq L(7,2)$ but not the same simple homotopy type.

Hurewicz '30: what if $\pi_1 M = 0$? Novikov '65: No — there is an S^4-bundle over S^2 which is not a product.

Borel conjecture

Borel '52 in a letter to Serre: what if $\pi_1(M) \neq 0$ but $\widetilde{M}$ is contractible?

Such manifolds are called *aspherical* indicating that all higher homotopy groups are trivial, very different from S^n.

But there are a lot of them in geometry (list coming soon).
Also Gromov: pick randomly a closed manifold; it is aspherical with probability 1.

Evidence after '52

Mostow '54: solvmanifolds

Mostow Rigidity Theorem: hyperbolic manifolds, more generally locally symmetric spaces

Farrell and Jones '90s: non-positively curved manifolds

special — Waldhausen '79: M^3 with accessible $\pi_1(M)$

Recent results

Known cases of the Farrell-Jones conjecture imply the Borel conjecture when $\pi_1(M)$ is from a class that includes hyperbolic groups, CAT(0)-groups, virtually linear groups (Lück et al. '07-'13), virtually solvable groups (Wegner '15), mapping class groups (Bartels/Bestvina '17).

Theorem (Carlsson/G.) The Borel conjecture is true when $\pi_1(M)$ has finite decomposition complexity. In particular, it is true when $\pi_1(M)$ has finite asymptotic dimension.

This includes all known aspherical manifolds except for the family constructed by Mark Sapir. This family is not known to have FDC.

A reformulation

Given M^n, the *structure set* $\mathcal{S}(M)$ is the set of equivalence classes of homotopy equivalences $f_1\colon M_1 \to M$ such that $f_2 \circ h = f_1$ for some homeomorphism h.

Now Borel says: $|\mathcal{S}(M)| = 1$ if M is aspherical.

Define $\mathcal{S}^h(M)$ in the same way as $\mathcal{S}(M)$ but with a weaker relation, using h-cobordisms instead of homeomorphisms.

Long exact sequence in algebraic surgery

Here $\Gamma = \pi_1(M)$.

$$\ldots \longrightarrow H_{n+1}(M, L(\mathbb{Z})) \xrightarrow{A_{L,n+1}} L_{n+1}(\mathbb{Z}[\Gamma]) \longrightarrow \mathcal{S}^h(M) \longrightarrow H_n(M, L(\mathbb{Z})) \xrightarrow{A_{L,n}} L_n(\mathbb{Z}[\Gamma]) \longrightarrow \ldots$$

Conjecture: all $A_{L,n}$ are isomorphisms, then $\mathcal{S}^h(M) = 1$.

Now h-cobordisms over M are in bijective correspondence with the *Whitehead group* $Wh(M)$.

So if $Wh(M) = 1$ then $\mathcal{S}^h(M) = \mathcal{S}(M)$.

Long exact sequence in algebraic K-theory

Again $\Gamma = \pi_1(M)$.

$$\dots \longrightarrow H_1(M, K(\mathbb{Z})) \xrightarrow{A_{K,1}} K_1(\mathbb{Z}[\Gamma]) \longrightarrow Wh(M) \\ \longrightarrow H_0(M, K(\mathbb{Z})) \xrightarrow{A_{K,0}} K_0(\mathbb{Z}[\Gamma]) \longrightarrow \dots$$

Conjecture: all $A_{K,n}$ are isomorphisms, then $Wh(M) = 1$.

The conjectures

The homomorphisms $A_{K,n}$ and $A_{L,n}$ are called *assembly maps*. We only want 4 of them to be isomorphisms: $A_{K,0}$, $A_{K,1}$, $A_{L,n+1}$, and $A_{L,n}$. But Hsiang (Warszawa ICM, '83) conjectured all of them are isomorphisms. Now this is known as the Isomorphism Conjecture.

Hsiang's *Isomorphism Conjecture*: if Γ is a group with a finite classifying space $K(\Gamma, 1)$ then all assembly maps are isomorphisms.

The injectivity portion is called the *Novikov Conjecture*.

What is K-theory?

Our results apply to L-theory and to K-theory, but I will only talk about K-theory.

I will treat K-theory as a black box. We only need to know that there is a machine that accepts an *additive category* A where $\oplus$ is defined and spits out a space. The homotopy groups of the space are the K-theory of A.

Example: if A is the category of finitely generated free or projective modules over some ring R then the machine gives $K(R)$.

When $R = \mathbb{Z}[\Gamma]$, $K(R)$ contains very important information for homeomorphism classification of manifolds with $\pi_1 = \Gamma$ in general.

Ideas from equivariant topology

Z topological space, Γ group acting on Z

Notation: $Z^\Gamma =$ fixed points, Z/Γ the orbit space

An issue: if $f\colon S \to T$ is an equivariant map and is a homotopy equivalence, the induced map on the fixed points $f^\Gamma\colon S^\Gamma \to T^\Gamma$ is NOT necessarily.

There is a repair.

Let's interpret Z^{Γ} as $Fun(point, Z)^{\Gamma}$.
The *homotopy fixed points* $Z^{h\Gamma}$ is the space defined as $Fun(E\Gamma, Z)^{\Gamma}$.
Here $E\Gamma$ is the universal free Γ space, the universal cover of a $K(\Gamma, 1)$.

Now for homotopy fixed points it is true that if $f\colon S \to T$ is an equivariant map and is a homotopy equivalence, the induced map on the homotopy fixed points $f^{h\Gamma}\colon S^{h\Gamma} \to T^{h\Gamma}$ is a homotopy equivalence.

The collapse $E\Gamma \to \mathit{point}$ induces $r\colon Z^\Gamma \to Z^{h\Gamma}$.
And we have a commutative square

$$\begin{array}{ccc} S^\Gamma & \xrightarrow{f^\Gamma} & T^\Gamma \\ \downarrow{\scriptstyle r_S} & & \downarrow{\scriptstyle r_T} \\ S^{h\Gamma} & \xrightarrow{f^{h\Gamma}} & T^{h\Gamma} \end{array}$$

Now I will describe how to use this to model the assembly map as a fixed point map and used homotopy fixed points to prove the Novikov Conjecture.

The claim is that there are S and T with Γ-actions built out of the manifold M and a map f so that $A_\bullet$ are induced from the space map $\alpha = f^\Gamma\colon S^\Gamma \to T^\Gamma$.

Let $X = \widetilde{M}$, so $M = X/\Gamma$.
$S = h^{lf}(X, K(\mathbb{Z}))$, the locally finite homology.
$T = K(X, \mathbb{Z})$, the bounded K-theory.

Facts:

- S^Γ is the homology of Γ, $H(B\Gamma, K(\mathbb{Z}))$.
- T^Γ is the K-theory of $\mathbb{Z}[\Gamma]$, $K(\mathbb{Z}[\Gamma])$.
- $\alpha = f^\Gamma \colon S^\Gamma \to T^\Gamma$ induces exactly the assembly maps $A_{K,n}$ on homotopy groups.
- $r_S \colon S^\Gamma \to S^{h\Gamma}$ is always an equivalence.

We just need to show there is an equivalence $f\colon h^{lf}(X, K(\mathbb{Z})) \to K(X, \mathbb{Z})$. This is enough to show that α is a split injection.

Ingredient #1. Locally finite homology

$h^{lf}(X)$ for a metric space X

$C_k^{lf} =$ infinite formal linear combinations $\sum n_\sigma \sigma$ where $\sigma \colon \Delta^n \to X$ are the usual singular simplices satisfying the following conditions:

1) for any compact $K \subset X$ there are only finitely many σ with $\mathrm{im}(\sigma) \cap K \neq \emptyset$ and $n_\sigma \neq 0$,

2) there is a uniform bound on diameters

Then C_*^{lf} is a chain complex, H_*^{lf} its homology

Properties:

- Functorial with respect to proper maps ($=$ pre-images of compact subsets are compact)

- Proper homotopy invariant

- $H_t^{lf}(\mathbb{R}^n) = 0$ for $t \neq n$ and $\mathbb{Z}$ for $t = n$.

- $H_*^{lf}(K) = H_*(K)$ for K compact.

Mayer-Vietoris

$$\begin{array}{ccc} \mathbb{R}^{n-1} = \mathbb{R}^n_- \cap \mathbb{R}^n_+ & \longrightarrow & \mathbb{R}^n_+ \\ \downarrow & & \downarrow \\ \mathbb{R}^n_- & \longrightarrow & \mathbb{R}^n \end{array}$$

induces a homotopy pushout on H^{lf}_*.

This suggests $H^{lf}_*(\mathbb{R}^n_+) = H^{lf}_*(\mathbb{R}^n_-) = 0$

Picture.

Ingredient #2. Bounded K-theory $K(X, R)$

After Pedersen/Weibel.

Objects = collections of choices F_x which are free finitely generated R-modules associated to each point x in X, with the requirement that only finitely many F_x are non-0 for x from a bounded subset S.

The module $F(X) = \bigoplus F_x$ can be given a filtration by $F(S) = \bigoplus_{x \in S} F_x$.

A homomorphism $f\colon F \to G$ is *bounded* if there is a number b so that for all subsets S we have $f(F(S)) \subset G(S[b])$, using the notation $S[b]$ for the b-enlargement of S.

This is an additive category $\rightsquigarrow$ $K(X, R)$

Again Mayer-Vietoris

$$\begin{array}{ccc} \mathbb{R}^{n-1} = \mathbb{R}^n_- \cap \mathbb{R}^n_+ & \longrightarrow & \mathbb{R}^n_+ \\ \downarrow & & \downarrow \\ \mathbb{R}^n_- & \longrightarrow & \mathbb{R}^n \end{array}$$

induces a homotopy pushout on K.

In fact can replace half-spaces by any pair of *antithetic* subsets U and V in any metric space X. This means for each number $K > 0$ there is a number $K' > 0$ so that

$$S[K] \cap T[K] \subset (S \cap T)[K'].$$

Then still true that

$$\begin{array}{ccc} S \cap T & \longrightarrow & T \\ \downarrow & & \downarrow \\ S & \longrightarrow & X = S \cup T \end{array}$$

induces a homotopy pushout on K. And on h^{lf}.

We have just proven the Novikov conjecture for tori T^n because their universal covers are $\mathbb{R}^n$.

Promote: excision on trees

Promote: excision on products $\rightsquigarrow$ products of trees

Promote: any subspace of products of trees

Promote: (Carlsson/G. '04) any metric space embedded in a finite product of trees via a uniformly expansive map $\equiv$ any metric space with finite asymptotic dimension (Dranishnikov)

Generalization: (Ramras/Tessera/Yu '16) any metric space with finite decomposition complexity

Surjectivity of the assembly

We specialize to M closed aspherical manifold, $\pi_1(M) = \Gamma$. We will denote $K_X = K(X, \mathbb{Z}) = K(\widetilde{M}, \mathbb{Z})$. Recall that we want to show that $r\colon K_X^{\Gamma} \to K_X^{h\Gamma}$ is a split injection. We will reuse the strategy for Novikov.

Embed M in $\mathbb{R}^n$ for large enough n, let N be the normal bundle.

We start with a suspension of the map we are interested in.

$$
\begin{array}{ccccc}
\Sigma^n K_X^{\Gamma} & \longrightarrow & \Sigma^n K_X^{h\Gamma} = \Sigma^n Fun(X, K_X)^{\Gamma} & \longrightarrow & h^{lf}(\widetilde{N}, K_X)^{\Gamma} \\
& \searrow & & & \downarrow \\
& & & & K(\widetilde{N} \times X)^{\Gamma} \\
& & & & \downarrow \\
& & & & \Sigma^n K_X^{\Gamma}
\end{array}
$$

Reality check

We can only prove this diagram is commutative when "K" is replaced by "G", the controlled G-theory in the title. But there is also a comparison theorem.

Theorem (Carlsson/G.) $K_X^\Gamma \cong G_X^\Gamma$ if Γ has straight FDC, property due to Dranishnikov/Zarichnyi which is weaker than FDC itself. This theorem also has an essentially coarse geometric proof. This property is called *coarse coherence*.

So Corollary The Borel conjecture is true for M with $\pi_1(M)$ that has FDC.

Slogan

"Borel $=$ Novikov $+$ coarse coherence"

It seems that now the generally believed "Novikov" is the bottleneck.

The honest statement

Theorem (Carlsson/G. '16)

Suppose R is a regular Noetherian ring of finite global dimension. (For example $\mathbb{Z}$.) Suppose Γ is a group with finite $BG = K(G,1)$ and has FDC.

Then the K-theoretic assembly α is an equivalence.

Remark: the geometric condition FDC or straight FDC contributes to both "Novikov" and "coherence" items.

With the the same assumptions one has the same conclusions in quadratic L-theory (Carlsson/G./Varisco, in progress).