# The Borel conjecture (through controlled G-theory)

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Topology is hard. A good idea is to geometrize your problem and exploit the geometry.

- Nikita Selinger

# Topological rigidity

Poincare conjecture: if  $M^n \simeq S^n$  then  $M^n \approx S^n$ .

Not true for arbitrary closed manifolds. Lens spaces  $L(7,1) \simeq L(7,2)$  but not the same simple homotopy type.

Hurewicz '30: what if  $\pi_1 M = 0$ ? Novikov '65: No — there is an  $S^4$ -bundle over  $S^2$  which is not a product.

#### Borel conjecture

Borel '52 in a letter to Serre: what if  $\pi_1(M) \neq 0$  but  $\widetilde{M}$  is contractible?

Such manifolds are called *aspherical* indicating that all higher homotopy groups are trivial, very different from  $S^n$ .

But there are a lot of them in geometry (list coming soon). Also Gromov: pick randomly a closed manifold; it is aspherical with probability 1.

### Evidence after '52

Mostow '54: solvmanifolds

Mostow Rigidity Theorem: hyperbolic manifolds, more generally locally symmetric spaces

Farrell and Jones '90s: non-positively curved manifolds

special — Waldhausen '79:  $M^3$  with accessible  $\pi_1(M)$ 

#### Recent results

Known cases of the Farrell-Jones conjecture imply the Borel conjecture when  $\pi_1(M)$  is from a class that includes hyperbolic groups, CAT(0)-groups, virtually linear groups (Lück et al. '07-'13), virtually solvable groups (Wegner '15), mapping class groups (Bartels/Bestvina '17).

Theorem (Carlsson/G.) The Borel conjecture is true when  $\pi_1(M)$  has finite decomposition complexity. In particular, it is true when  $\pi_1(M)$  has finite asymptotic dimension.

This includes all known aspherical manifolds except for the family constructed by Mark Sapir. This family is not known to have FDC.

# A reformulation

Given  $M^n$ , the structure set  $\mathcal{S}(M)$  is the set of equivalence classes of homotopy equivalences  $f_1: M_1 \to M$  such that  $f_2 \circ h = f_1$  for some homeomorphism h.

Now Borel says: |S(M)| = 1 if M is aspherical.

Define  $S^h(M)$  in the same way as S(M) but with a weaker relation, using h-cobordisms instead of homeomorphisms.

Long exact sequence in algebraic surgery Here  $\Gamma = \pi_1(M)$ .

$$\dots \longrightarrow H_{n+1}(M, L(\mathbb{Z})) \xrightarrow{A_{L,n+1}} L_{n+1}(\mathbb{Z}[\Gamma]) \longrightarrow S^{h}(M)$$
$$\longrightarrow H_{n}(M, L(\mathbb{Z})) \xrightarrow{A_{L,n}} L_{n}(\mathbb{Z}[\Gamma]) \longrightarrow \dots$$

Conjecture: all  $A_{L,n}$  are isomorphisms, then  $S^h(M) = 1$ .

Now h-cobordisms over M are in bijective correspondence with the Whitehead group Wh(M).

So if Wh(M) = 1 then  $S^h(M) = S(M)$ .

Long exact sequence in algebraic K-theory Again  $\Gamma = \pi_1(M)$ .

$$\dots \longrightarrow H_1(M, K(\mathbb{Z})) \xrightarrow{A_{K,1}} K_1(\mathbb{Z}[\Gamma]) \longrightarrow Wh(M) \longrightarrow H_0(M, K(\mathbb{Z})) \xrightarrow{A_{K,0}} K_0(\mathbb{Z}[\Gamma]) \longrightarrow \dots$$

Conjecture: all  $A_{K,n}$  are isomorphisms, then Wh(M) = 1.

#### The conjectures

The homomorphisms  $A_{K,n}$  and  $A_{L,n}$  are called *assembly maps*. We only want 4 of them to be isomorphisms:  $A_{K,0}$ ,  $A_{K,1}$ ,  $A_{L,n+1}$ , and  $A_{L,n}$ . But Hsiang (Warszawa ICM, '83) conjectured all of them are isomorphisms. Now this is known as the Isomorphism Conjecture.

Hsiang's *Isomorphism Conjecture*: if  $\Gamma$  is a group with a finite classifying space  $K(\Gamma, 1)$  then all assembly maps are isomorphisms.

The injectivity portion is called the Novikov Conjecture.

# What is K-theory?

Our results apply to L-theory and to K-theory, but I will only talk about K-theory.

I will treat K-theory as a black box. We only need to know that there is a machine that accepts an *additive category* A where  $\oplus$  is defined and spits out a space. The homotopy groups of the space are the K-theory of A.

Example: if A is the category of finitely generated free or projective modules over some ring R then the machine gives K(R).

When  $R = \mathbb{Z}[\Gamma]$ , K(R) contains very important information for homeomorphism classification of manifolds with  $\pi_1 = \Gamma$  in general.

# Ideas from equivariant topology

Z topological space,  $\Gamma$  group acting on Z

Notation:  $Z^{\Gamma}$  = fixed points,  $Z/\Gamma$  the orbit space

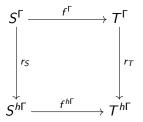
An issue: if  $f: S \to T$  is an equivariant map and is a homotopy equivalence, the induced map on the fixed points  $f^{\Gamma}: S^{\Gamma} \to T^{\Gamma}$  is NOT necessarily.

There is a repair.

Let's interpret  $Z^{\Gamma}$  as  $Fun(point, Z)^{\Gamma}$ . The homotopy fixed points  $Z^{h\Gamma}$  is the space defined as  $Fun(E\Gamma, Z)^{\Gamma}$ . Here  $E\Gamma$  is the universal free  $\Gamma$  space, the universal cover of a  $K(\Gamma, 1)$ .

Now for homotopy fixed points it is true that if  $f: S \to T$  is an equivariant map and is a homotopy equivalence, the induced map on the homotopy fixed points  $f^{h\Gamma}: S^{h\Gamma} \to T^{h\Gamma}$  is a homotopy equivalence.

The collapse  $E\Gamma \rightarrow point$  induces  $r: Z^{\Gamma} \rightarrow Z^{h\Gamma}$ . And we have a commutative square



Now I will describe how to use this to model the assembly map as a fixed point map and used homotopy fixed points to prove the Novikov Conjecture.

The claim is that there are S and T with  $\Gamma$ -actions built out of the manifold M and a map f so that  $A_{\bullet}$  are induced from the space map  $\alpha = f^{\Gamma} : S^{\Gamma} \to T^{\Gamma}$ .

Let  $X = \widetilde{M}$ , so  $M = X/\Gamma$ .  $S = h^{lf}(X, K(\mathbb{Z}))$ , the locally finite homology.  $T = K(X, \mathbb{Z})$ , the bounded K-theory.

Facts:

- $S^{\Gamma}_{-}$  is the homology of  $\Gamma$ ,  $H(B\Gamma, K(\mathbb{Z}))$ .
- $T^{\Gamma}$  is the K-theory of  $\mathbb{Z}[\Gamma]$ ,  $K(\mathbb{Z}[\Gamma])$ .

-  $\alpha = f^{\Gamma} \colon S^{\Gamma} \to T^{\Gamma}$  induces exactly the assembly maps  $A_{K,n}$  on homotopy groups.

-  $r_S \colon S^{\Gamma} \to S^{h\Gamma}$  is always an equivalence.

We just need to show there is an equivalence  $f: h^{lf}(X, K(\mathbb{Z})) \to K(X, \mathbb{Z})$ . This is enough to show that  $\alpha$  is a split injection.

### Ingredient #1. Locally finite homology

 $h^{lf}(X)$  for a metric space X

 $C_k^{lf}$  = infinite formal linear combinations  $\sum n_\sigma \sigma$  where  $\sigma \colon \Delta^n \to X$  are the usual singular simplices satisfying the following conditions:

1) for any compact  $K \subset X$  there are only finitely many  $\sigma$  with  $im(\sigma) \cap K \neq \emptyset$  and  $n_{\sigma} \neq 0$ ,

2) there is a uniform bound on diameters

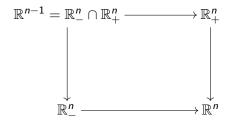
Then  $C_*^{lf}$  is a chain complex,  $H_*^{lf}$  its homology

Properties:

- Functorial with respect to proper maps (= pre-images of compact subsets are compact)

- Proper homotopy invariant
- $H_t^{lf}(\mathbb{R}^n) = 0$  for  $t \neq n$  and  $\mathbb{Z}$  for t = n.
- $H_*^{lf}(K) = H_*(K)$  for K compact.

Mayer-Vietoris



induces a homotopy pushout on  $H_*^{lf}$ . This suggests  $H_*^{lf}(\mathbb{R}^n_+) = H_*^{lf}(\mathbb{R}^n_-) = 0$ 

Picture.

## Ingredient #2. Bounded K-theory K(X, R)

After Pedersen/Weibel.

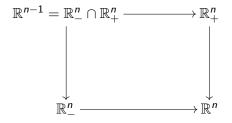
Objects = collections of choices  $F_x$  which are free finitely generated R-modules associated to each point x in X, with the requirement that only finitely many  $F_x$  are non-0 for x from a bounded subset S.

The module  $F(X) = \bigoplus F_x$  can be given a filtration by  $F(S) = \bigoplus_{x \in S} F_x$ .

A homomorphism  $f: F \to G$  is *bounded* if there is a number b so that for all subsets S we have  $f(F(S)) \subset G(S[b])$ , using the notation S[b] for the *b*-enlargement of S.

This is an additive category  $\rightsquigarrow K(X, R)$ 

Again Mayer-Vietoris

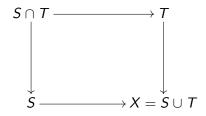


induces a homotopy pushout on K.

In fact can replace half-spaces by any pair of *antithetic* subsets U and V in any metric space X. This means for each number K > 0 there is a number K' > 0 so that

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S[K] \cap T[K] \subset (S \cap T)[K'].
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Then still true that



induces a homotopy pushout on K. And on  $h^{lf}$ .

We have just proven the Novikov conjecture for tori  $T^n$  because their universal covers are  $\mathbb{R}^n$ .

Promote: excision on trees

Promote: excision on products ~> products of trees

Promote: any subspace of products of trees

Promote: (Carlsson/G. '04) any metric space embedded in a finite product of trees via a uniformly expansive map  $\equiv$  any metric space with finite asymptotic dimension (Dranishnikov)

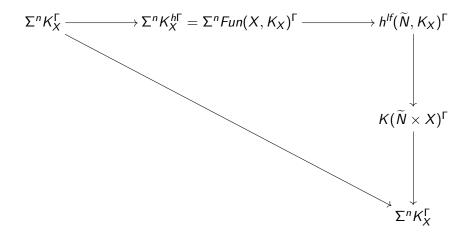
Generalization: (Ramras/Tessera/Yu '16) any metric space with finite decomposition complexity

### Surjectivity of the assembly

We specialize to M closed aspherical manifold,  $\pi_1(M) = \Gamma$ . We will denote  $K_X = K(X, \mathbb{Z}) = K(\widetilde{M}, \mathbb{Z})$ . Recall that we want to show that  $r: K_X^{\Gamma} \to K_X^{h\Gamma}$  is a split injection. We will reuse the strategy for Novikov.

Embed *M* in  $\mathbb{R}^n$  for large enough *n*, let *N* be the normal bundle.

We start with a suspension of the map we are interested in.



### Reality check

We can only prove this diagram is commutative when "K" is replaced by "G", the controlled G-theory in the title. But there is also a comparison theorem.

Theorem (Carlsson/G.)  $K_X^{\Gamma} \cong G_X^{\Gamma}$  if  $\Gamma$  has straight FDC, property due to Dranishnikov/Zarichnyi which is weaker than FDC itself. This theorem also has an essentially coarse geometric proof. This property is called *coarse coherence*.

So Corollary The Borel conjecture is true for M with  $\pi_1(M)$  that has FDC.

# Slogan

"Borel = Novikov + coarse coherence"

It seems that now the generally believed "Novikov" is the bottleneck.

#### The honest statement

Theorem (Carlsson/G. '16)

Suppose *R* is a regular Noetherian ring of finite global dimension. (For example  $\mathbb{Z}$ .) Suppose  $\Gamma$  is a group with finite BG = K(G, 1) and has FDC.

Then the K-theoretic assembly  $\alpha$  is an equivalence.

Remark: the geometric condition FDC or straight FDC contributes to both "Novikov" and "coherence" items.

With the same assumptions one has the same conclusions in quadratic *L*-theory (Carlsson/G./Varisco, in progress).